

GEOMETRIC CRITERIA FOR OVERTWISTEDNESS

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ABSTRACT. We establish geometric criteria to decide whether a contact manifold is overtwisted. Starting with the original definition, we first relate the different overtwisted disks (D^{2n}, ξ_{ot}) in each dimension and show that a manifold is overtwisted if the Legendrian unknot is loose. Then we characterize overtwistedness in terms of open book decompositions and provide several applications.

1. INTRODUCTION

A contact structure on a $(2n - 1)$ -dimensional smooth manifold Y is a maximally non-integrable codimension 1 distribution ξ . Recently a special class of contact structures has been introduced [3] in any dimension: the overtwisted contact structures. Generalizing the original definition and results in the 3-dimensional case [15], it is shown in [3] that overtwisted contact structures satisfy a parametric h -principle, i.e. their classification up to isotopy coincides with the classification of homotopy classes of almost contact structures. This classification then becomes a strictly algebraic topological problem which can be solved via obstruction theory. The definition of overtwisted contact structures is reviewed in Section 3.

1.1. The main theorem. Though the result in [3] demonstrates the existence of overtwisted contact structures homotopic to any almost contact structure, a drawback to the existence proof is that the construction is fairly non-explicit. There is a lack of examples of closed overtwisted contact manifolds of dimension $2n - 1 > 3$, and the techniques used in [3] give no criterion in order to show that a given manifold is overtwisted, other than an direct application of the definition. The main result of this paper gives a number of equivalent conditions for overtwistedness.

Theorem 1.1. *Let (Y, ξ) be a contact manifold of dimension $2n - 1 > 3$. Choose a contact form α_{ot} on \mathbb{R}^3 which defines an overtwisted contact structure. Then there is a constant $R \in \mathbb{R}^+$ depending only on α_{ot} and n , so that the following conditions are equivalent:*

1. (Y, ξ) is overtwisted.
- 2a. There is a contact embedding of $(\mathbb{R}^3 \times \mathbb{C}^{n-2}, \ker\{\alpha_{ot} + \lambda_{st}\})$ into (Y, ξ) .
- 2b. There is a contact embedding of $(\mathbb{R}^3 \times D^{2n-4}(R), \ker\{\alpha_{ot} + \lambda_{st}\})$ into (Y, ξ) .
- 3a. The standard Legendrian unknot $\Lambda_0 \subseteq Y$ is loose.
- 3b. (Y, ξ) contains a small plastikstufe with spherical core and trivial rotation.
4. Y can be obtained by $(+1)$ -surgery on some loose Legendrian $\Lambda \subseteq Y'$, for some (Y', ξ') .
5. There is an open book compatible with (Y, ξ) which is a negative stabilization.

In the statement of Theorem 1.1 above, λ_{st} is the standard Liouville form on both the disk $D^{2n-4}(R)$ and \mathbb{C}^{n-2} , defined by

$$\lambda_{st} = \frac{1}{2} \sum_{i=1}^{n-2} (x_i dy_i - y_i dx_i) = \frac{1}{2} \sum_{i=1}^{n-2} r_i^2 d\theta_i.$$

The standard Legendrian unknot $\Lambda_0 \subseteq (\mathbb{R}^{2n-1}, \xi_{st})$ is defined to be

$$\Lambda_0 = \{y_i = 0 : i = 1, \dots, n\} \cap S^{2n-1} \subseteq (\mathbb{R}^{2n-1}, \xi_{st}) = (S^{2n-1}, \xi_{st}) \setminus \{\text{point}\} \subseteq \mathbb{C}^n.$$

The standard Legendrian unknot $\Lambda_0 \subseteq (Y, \xi)$ is defined by the inclusion of a Darboux chart $\Lambda_0 \subseteq (\mathbb{R}^{2n-1}, \xi_{\text{st}}) \subseteq (Y, \xi)$, all of which are isotopic. The concept of *loose* Legendrians was first studied in [33], and see also [11, 34].

The *plastikstufe* is an n -dimensional submanifold $\mathcal{P} \subseteq (Y, \xi)$ so that the contact structure ξ is equivalent to $D_{\text{ot}}^2 \times \{p = 0\} \subseteq (\mathbb{R}_{\text{ot}}^3 \times T^*Q, \ker\{\alpha_{\text{ot}} + \lambda_{\text{Liouville}}\})$ near \mathcal{P} , where Q is a closed manifold, called the *core* of \mathcal{P} , and $\{p = 0\}$ is the zero section $Q \subseteq T^*Q$. The plastikstufe was first defined in [36] and shown there to be an obstruction to symplectic fillability. See also [34] for the definitions of “small” and “trivial rotation”, and how the plastikstufe relates to loose Legendrians.

(+1)-*surgery* is implicit in the theory of Weinstein handle attachments [11]; it is the surgery induced by attaching a *concave* handle to a compact piece of the symplectization. In the higher dimensional case it was studied in depth in [1]. In particular, the implication $1 = 4$ is stated there as Conjecture 9.16.

Open books which are compatible with (Y, ξ) appear in the Giroux correspondance between open books and contact structures [12, 24]. In order for the fifth characterization in Theorem 1.1 and this discussion to apply, we suppose that (Y, ξ) is closed. An appropriate open book decomposition of the manifold Y determines a contact structure ξ , and every contact manifold (Y, ξ) admits such an adapted open book decomposition. This adapted open book can be positively or negatively stabilized. The resulting open books induce two contact structures ξ_+ and ξ_- on Y . The positive stabilization (Y, ξ_+) is contactomorphic to (Y, ξ) , but (Y, ξ_-) typically is not. In particular, the negative stabilization of a contact structure was known to have vanishing symplectic field theory [5, 6]. In particular, using [6, Theorem 1.3], Theorem 1.1 then implies that the contact homology of an overtwisted contact manifold vanishes.

See Section 2 for a more detailed discussion on the notions used in Theorem 1.1.

1.2. The argument for Theorem 1.1. Let us detail the logic of the proof for the implications in Theorem 1.1. The entire argument is an induction in dimension. The existence h-principle in [3] readily implies that if (Y, ξ) is overtwisted then $2a$, $2b$, $3a$ and $3b$ hold.

- The equivalence $1 = 2a = 2b$: since $2a$ implies $2b$, it suffices to show that $2b$ implies 1. This is the content of Theorem 3.1, proved in Section 3.
- The equivalence $1 = 3a = 3b$: the implication $3a \Rightarrow 1$ is proven in Section 4 as a consequence of Theorem 4.5. The main ingredient is Lemma 4.2, which is where the inductive hypothesis is used. Note that $3b \Rightarrow 3a$ follows from [34, Theorem 1.1].
- The equivalence $1 = 4$: it is proven in Section 5 where we show $4 \Rightarrow 3b$.
- The equivalence $1 = 5$: it is detailed in Section 6, the argument proves $5 \Rightarrow 3a$.

Remark 1.2. There are also a number of direct implications which can give alternative proofs of some of the arguments. For example, $\{1 = 4\}$ can be proved directly from $\{1 = 2b\}$ without reference to $\{1 = 3b\}$ [19]. It could also be proved using $\{1 = 5\}$, together with approximately holomorphic techniques which realize $\Lambda \subseteq (Y, \xi)$ as an exact Lagrangian on a Weinstein page of a supporting open book [8]. Likely there are other possible arguments directly connecting the various criteria in Theorem 1.1.

1.3. Organization of the article. There are seven sections in the article. Section 3 to 6 contain the argument for Theorem 1.1: Section 3 proves $1 = 2a = 2b$, Section 4 shows $1 = 3a = 3b$, Section 5 proves $1 = 4$ and Section 6 concludes $1 = 5$. Finally, Section 7 gives some applications of Theorem 1.1 to contact squeezing results and constructions of Weinstein cobordisms. Each of these sections also contains results that can be of interest on their own.

In particular, the connection developed in Section 6 is relevant for higher dimensional contact topology.

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2. CONTACT GEOMETRY BACKGROUND

In this section we review a number of relevant definitions and results in high dimensional contact topology which are used along the article. These preliminaries are necessary for an understanding of Theorem 1.1 and its proof.

2.1. Loose Legendrians. The notion of a loose Legendrian appears in the equivalence $1 = 3a$. Let $B^3 \subseteq (\mathbb{R}^3, \xi_{\text{st}})$ be a round ball in a contact Darboux chart and let $\Lambda_0 \subseteq (\mathbb{R}^3, \xi_{\text{st}})$ be a stabilized Legendrian arc as seen in Figure 1. Consider a closed manifold Q and a neighborhood $\mathcal{O}p(Z) \subseteq T^*Q$ of the zero section $Z \subseteq T^*Q$. Then the smooth submanifold $\Lambda_0 \times Z \subseteq (B^3 \times \mathcal{O}p(Z), \ker(\alpha_{\text{st}} + \lambda_{\text{st}}))$ is Legendrian submanifold.

Definition 2.1. Let Q be a closed manifold and $Z \subseteq T^*Q$ the zero section. The pair $(B^3 \times \mathcal{O}p(Z), \Lambda_0 \times Z)$ endowed with the contact structure $\ker(\alpha_{\text{st}} + \lambda_{\text{st}})$ is said to be a *loose chart*.

Let $\Lambda \subseteq (Y, \xi)$ be a Legendrian in a contact manifold with $\dim(Y) \geq 5$. The Legendrian Λ is *loose* in (Y, ξ) if there is an open set $V \subseteq Y$ so that $(V, V \cap \Lambda)$ is contactomorphic to a loose chart.

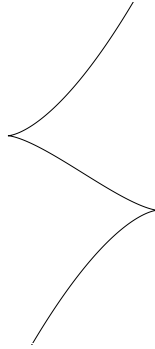


FIGURE 1. The front projection of a stabilized Legendrian arc.

Loose Legendrians were first defined and classified in [33]. The definition presented above is slightly different than the definition introduced there, though it is equivalent (see [34, Section 4.2]). The following property will be most useful for us.

Theorem 2.2 ([33]). *Let Λ_0 be a loose Legendrian, and let $f_t : \Lambda \rightarrow Y$ be a smooth isotopy so that f_0 is the inclusion map, and f_1 is unconstrained other than being a smooth embedding. Then there is a Legendrian isotopy $g_t : \Lambda \rightarrow Y$ which is C^0 -close to f_t .*

This is used in Proposition 2.8 below, which in turn is needed in Theorems 4.4 and 7.6.

2.2. The plastikstufe. We provide details on the definition of the plastikstufe, first introduced in the article [36]. Let $\mathcal{O}p(D_{\text{ot}}^2) \subseteq (\mathbb{R}^3, \xi_{\text{ot}})$ be a contact neighborhood of an overtwisted disk for any overtwisted contact structure $\xi_{\text{ot}} = \ker \alpha_{\text{ot}}$.

Definition 2.3. Let Q be a closed manifold and $\mathcal{O}p(Z) \subseteq T^*Q$ a neighborhood of the zero section. The contact manifold $(\mathcal{O}p(D_{\text{ot}}^2 \times \mathcal{O}p(Z)), \ker(\alpha_{\text{ot}} + \lambda_{\text{st}}))$ is said to be a *plastikstufe*. The submanifold $Q \subseteq \mathcal{O}p(D_{\text{ot}}^2) \times \mathcal{O}p(Z)$ is the *core* of the plastikstufe.

Definition 2.4. A plastikstufe $\mathcal{O}p(D_{\text{ot}}^2) \times \mathcal{O}p(Z) \subseteq (Y, \xi)$ is said to be *small* if it is contained in a smooth ball in Y . Let $\Lambda_0 \subseteq \mathcal{O}p(D_{\text{ot}}^2)$ be an open leaf of the characteristic foliation of the overtwisted disk. A small plastikstufe $(\mathcal{O}p(D_{\text{ot}}^2) \times \mathcal{O}p(Z), \ker(\alpha_{\text{ot}} + \lambda_{\text{st}}))$ has *trivial rotation* if the open Legendrian submanifold $\Lambda_0 \times Z$ has trivial rotation class.

Note that, for a small plastikstufe, the rotation class of the Legendrian $\Lambda_0 \times Z$ is well defined since the hyperplane field ξ has a unique framing on the smooth ball up to homotopy. Observe that in the case $Q = S^{n-2}$, then a plastikstufe $\mathcal{O}p(D_{\text{ot}}^2) \times \mathcal{O}p(Z) \subseteq (Y, \xi)$ is both small and has trivial rotation if and only if $\Lambda_0 \times Z$, which is a Legendrian annulus $[0, 1] \times S^{n-2}$, can be included into a Legendrian disk. Then this disk defines a smooth ball containing the plastikstufe, and since a Legendrian disk has a unique framing it induces a trivial framing on its boundary collar.

The following theorem from [34] gives in particular the implication $3b \Rightarrow 3a$.

Theorem 2.5 ([34]). *Let $\mathcal{PS} \subseteq (Y, \xi)$ be a small plastikstufe with spherical core and trivial rotation. Suppose $\Lambda \subseteq (Y, \xi)$ is a Legendrian disjoint from \mathcal{PS} , then the Legendrian Λ is loose.*

2.3. Weinstein manifolds. We review some basic definitions for Weinstein manifolds, see [11] for a thorough treatment of the topic.

Definition 2.6. A *Weinstein cobordism* is a triple (W, λ, f) , where $(W, d\lambda)$ is a compact symplectic manifold with boundary, and $f : W \rightarrow [0, 1]$ is a Morse function such that $\partial W = \partial_- W \cup \partial_+ W = f^{-1}(0) \cup f^{-1}(1)$. We also require that the vector field V_λ symplectic dual to the Liouville form λ (defined by $V_\lambda \lrcorner d\lambda = \lambda$) is gradient-like for f .

From the definition it follows that $\lambda|_{f^{-1}(c)}$ is a contact form on the submanifold $f^{-1}(c)$ for any regular value $c \in [0, 1]$. The descending manifold D_p^k associated to any critical point p of f satisfies $\lambda|_{D_p^k} = 0$. In particular the submanifold D_p^k is isotropic and thus

$$\text{ind}(p) = k \leq n = \frac{1}{2} \dim W.$$

Critical points with index strictly less than n are called *subcritical*, and a *subcritical* Weinstein cobordism (W, λ, f) is one where all critical points of f are subcritical.

In cases $c \in [0, 1]$ is a regular value, $\Lambda_p^c = D_p^k \cap f^{-1}(c)$ is an isotropic submanifold of the contact manifold $(f^{-1}(c), \ker \lambda)$. In case $c \in [0, 1]$ is a critical value with a unique critical point $p \in W$, the Weinstein cobordism $(f^{-1}([c - \varepsilon, c + \varepsilon]), \lambda, f)$ is determined, up to homotopy through Weinstein structures, by the contact manifold $(f^{-1}(c - \varepsilon), \ker \lambda)$ and the (parametrized) isotropic submanifold $\Lambda_p^{c-\varepsilon}$. Hence the contact manifold $(f^{-1}(c + \varepsilon), \ker \lambda)$ is determined up to contactomorphism, and it is said to be obtained from $(f^{-1}(c - \varepsilon), \ker \lambda)$ by *contact surgery along* the isotropic sphere $\Lambda_p^{c-\varepsilon}$. Notice that $(f^{-1}(c - \varepsilon) \setminus \Lambda_p^{c-\varepsilon}, \ker \lambda)$ has a natural contact inclusion into $(f^{-1}(c + \varepsilon), \ker \lambda)$, defined by the flow of the gradient-like vector field V_λ . We refer to the monograph [11] for proofs of these statements and a more complete discussion of Weinstein handle attachments.

In case $c \in [0, 1]$ is a critical value of f with a unique critical point p of index n then $\Lambda_p^{c-\varepsilon} \subseteq (f^{-1}(c - \varepsilon), \ker \lambda)$ is a Legendrian submanifold. If this Legendrian is loose, we

say that p is a *flexible* critical point (in dimension $2n = 4$, p is called flexible if $\Lambda_p^{c-\varepsilon}$ has overtwisted complement, though we do not discuss this dimension anywhere in this paper).

Definition 2.7. A Weinstein cobordism (W, λ, f) is said to be *flexible* if every critical point of f is either subcritical or flexible.

In particular, every subcritical Weinstein cobordism is flexible. Flexible Weinstein manifolds are completely classified, see [11, Chapter 14]. The following proposition is used to prove Theorem 1.1:

Proposition 2.8. *Let (W, λ, f) be a flexible Weinstein cobordism so that $(\partial_- W, \ker \lambda)$ is an overtwisted contact manifold. Then the contact manifold $(\partial_+ W, \ker \lambda)$ is overtwisted.*

Proof. Split the cobordism (W, λ, f) into cobordisms with a single critical point

$$W = f^{-1}([0, c_1]) \cup \dots \cup f^{-1}([c_s, 1]), \quad \text{for } 0 < c_1 < \dots < c_s < 1.$$

The resulting attaching spheres $\Lambda_j \subseteq (f^{-1}(c_j), \ker \lambda)$ are either subcritical or loose Legendrians submanifolds. We show by induction that each contact manifold $(f^{-1}(c_j), \ker \lambda)$ is overtwisted. The $j = 0$ case follows from the fact that $(\partial_- W, \ker \lambda)$ is overtwisted, and the case $j = s$ case implies the result. The contact manifold $(f^{-1}(c_{j+1}), \ker \lambda)$ is obtained from $(f^{-1}(c_j), \ker \lambda)$ by a single Weinstein surgery along the isotropic sphere Λ_j , and any smooth isotopy of Λ_j can be C^0 -approximated by a contact isotopy. Indeed, if Λ_j is subcritical this follows from the h -principle for subcritical isotropic submanifolds [25], and if Λ_j is a loose Legendrian this is Theorem 2.2. In particular, we can find a contact isotopy which makes the attachig isotropic sphere Λ_j disjoint from any overtwisted disk in $(f^{-1}(c_j), \ker \lambda)$. \square

Finally, we define a connect sum operation of Weinstein cobordisms. Let (W_1, λ_1, f_1) , (W_2, λ_2, f_2) be two Weinstein cobordisms with non-empty negative boundary, and choose two points $p_i \in \partial_- W_i$ which are not in the descending manifold of any critical point. Let γ_i be the image curves of the points p_i by the flow of the gradient-like vector fields V_{λ_i} , so $\gamma_i \subseteq W_i$ are two curves which intersect transversely every level set of their corresponding ambient cobordisms exactly once. We define the connected sum cobordism

$$W_1 \# W_2 = (W_1 \setminus \mathcal{O}p(\gamma_1)) \cup (W_2 \setminus \mathcal{O}p(\gamma_2)),$$

where the union glues a collar neighborhood of $\partial \mathcal{O}p(\gamma_1)$ to a collar neighborhood of $\partial \mathcal{O}p(\gamma_2)$ with a map that pulls back λ_2 to λ_1 and f_2 to f_1 (one can verify that such a map exists). The manifold $W_1 \# W_2$ inherits a Weinstein structure $(W_1 \# W_2, \lambda, f)$, the critical set of f is the union of critical sets of f_1 and f_2 , and every regular level set $(f^{-1}(c), \ker \lambda)$ is contactomorphic to the contact connected sum $(f_1^{-1}(c), \ker \lambda_1) \# (f_2^{-1}(c), \ker \lambda_2)$. The Weinstein manifold $(W_1 \# W_2, \lambda, f)$ is the *vertical connected sum* of (W_1, λ_1, f_1) and (W_2, λ_2, f_2) . This operation is used in [34, Section 5] to construct contactomorphisms using the flexible Weinstein h -cobordism theorem [11], our use of the vertical connected sum in Sections 4 and 7 is fairly similar.

2.4. Open book decompositions. Let (W, λ) be a Liouville domain, i.e. an exact symplectic manifold with the vector field V_λ outwardly transverse to ∂W , and $\varphi : W \rightarrow W$ a compactly supported exact symplectomorphism, so that $\varphi^* \lambda = \lambda + dh$ for some compactly supported function $h \in C_c^\infty(W)$. The triple (W, λ, φ) is an (abstract) *open book decomposition*. An open book decomposition (W, λ, φ) canonically defines a contact manifold (Y, ξ) [13, 24], by considering

$$Y = W \times [0, 1] / (x, 1) \sim (\varphi(x), 0) \cup_{\partial W \times S^1} \partial W \times D^2$$

$$\xi = \ker ((\lambda + K d\theta + \theta dh) \cup (\lambda|_{\partial W} + K r^2 d\theta)).$$

for a sufficiently large $K \in \mathbb{R}^+$. We write $(Y, \xi) = \text{OB}(W, \lambda, \varphi)$ to denote this relationship, and say that (Y, ξ) is *compatible with* or *supported by* the open book (W, λ, φ) . Notice that $\text{OB}(W, \lambda, \varphi) = \text{OB}(W, \lambda, \psi \circ \varphi \circ \psi^{-1})$ for any symplectomorphism ψ .

Open book decompositions are particularly useful in contact topology in light of E. Giroux's existence theorem [24].

Theorem 2.9 ([24]). *Every contact manifold (Y, ξ) can be presented as $(Y, \xi) = \text{OB}(W, \lambda, \varphi)$, and there exists a Morse function $f : W \rightarrow [0, 1]$ so that (W, λ, f) is a Weinstein manifold.*

Let (W, λ) be a Liouville manifold, and suppose it contains a (parametrized) Lagrangian sphere $L \subseteq (W, \lambda)$. We denote the *Dehn–Seidel* twist [39] around L by $\tau_L \in \text{Samp}_c(W)$.

Since L is an exact Lagrangian, L defines a Legendrian Λ in the contact manifold $\text{OB}(W, \lambda, \varphi)$ by integrating the exact form $\lambda|_L$. We denote this dependency by $(Y, \xi, \Lambda) = \text{OB}(W, \lambda, \varphi, L)$. The conjugation invariance above now reads

$$\text{OB}(W, \lambda, \varphi, L) = \text{OB}(W, \lambda, \psi \circ \varphi \circ \psi^{-1}, \psi(L))$$

which can be verified by considering Λ as being near the page $\theta = 0$.

The following proposition relates Dehn twists to contact surgery [28]:

Proposition 2.10. *Suppose that $(Y, \xi, \Lambda) = \text{OB}(W, \lambda, \varphi, L)$, then the contact manifold $\text{OB}(W, \lambda, \varphi \circ \tau_L)$ is obtained from (Y, ξ) by contact surgery along Λ .*

Note that both the mapping class $[\tau_L] \in \pi_0 \text{Samp}_c(W)$ and the contact surgery along Λ depend on parametrizations $S^{n-1} \cong L$ and $S^{n-1} \cong \Lambda$ respectively, which is often non-canonical. The diffeomorphism $\Lambda \cong L$ is however canonically given by projection to the page (W, λ) .

Consider a Lagrangian disk $D \subseteq (W, \lambda)$ with Legendrian boundary $\partial D \subseteq (\partial W, \ker \lambda)$ and attach a Weinstein handle to (W, λ) along the Legendrian sphere ∂D , obtaining a new Weinstein manifold $(W \cup H, \lambda')$. This manifold contains a Lagrangian sphere S , whose lower hemisphere is D and whose upper hemisphere is the core of the handle H . The open book $(W \cup H, \lambda', \varphi \circ \tau_L)$ is the *positive stabilization* of (W, λ, φ) along D , and $(W \cup H, \lambda', \varphi \circ \tau_L^{-1})$ is the *negative stabilization* of (W, λ, φ) along D .

Both the positive and the negative stabilization of an open book decomposition can be described as a contact connected sum. This description is the content of the following theorem.

Theorem 2.11 (E. Giroux). *Let $(Y, \xi) = \text{OB}(W, \lambda, \varphi)$ be a contact manifold, $D \subseteq (W, \lambda)$ any Lagrangian disk with Legendrian boundary $\partial D \subseteq (\partial W, \ker \lambda)$, and consider the contact structure $(S^{2n-1}, \xi_-) = \text{OB}(T^*S^{n-1}, \tau^{-1})$.*

Then the positive and negative stabilizations of (W, λ, φ) along D are diffeomorphic to Y . The positive stabilization is contactomorphic to (Y, ξ) , and the negative stabilization is contactomorphic to the contact connected sum $(Y \# S^{2n-1}, \xi \# \xi_-)$.

3. THICK NEIGHBORHOODS OF OVERTWISTED SUBMANIFOLDS

In this section we begin the proof of Theorem 1.1 with the equivalence $1 = 2$. Since the implications $1 \Rightarrow 2a \Rightarrow 2b$ certainly hold, it suffices to prove $2b \Rightarrow 1$. This is the content of the following theorem.

Theorem 3.1. *Let $(Y^{2n-1}, \ker \alpha_{ot})$ be an overtwisted contact structure. Then for sufficiently large R , the contact manifold $(Y \times D^2(R), \ker(\alpha_{ot} + \lambda_{st}))$ is overtwisted.*

The radius $R \in \mathbb{R}^+$ that appears in the statement depends on the choice of contact form α_{ot} for the contact manifold $(Y, \ker \alpha_{ot})$. Theorem 3.1 and its proof require some preliminaries, including the definition of the overtwisted disk [3]. This definition is reviewed in Subsection

3.1. Theorem 3.1 is proven in Subsection 3.2 for the case $n = 2$ and in Subsection 3.3 for $n \geq 3$. This distinction is not essential but it hopefully contributes to a better understanding of the result.

3.1. Overtwisted Disks. Let us review the definition of an overtwisted disk in any dimension[3]. Consider cylindrical coordinates

$$(z, u_1, \dots, u_{n-2}, \varphi_1, \dots, \varphi_{n-2}) \in \mathbb{R}^{2n-3} = \mathbb{R} \times (\mathbb{R}^2)^{n-2}$$

with each pair $(\sqrt{u_i}, \varphi_i) \in \mathbb{R}^2$ being polar coordinates. The standard contact structure $(\mathbb{R}^{2n-3}, \xi_{\text{st}})$ is given by the kernel of the 1-form

$$\alpha_{\text{st}} = dz + \sum_{i=1}^{n-2} u_i d\varphi_i = dz + u d\varphi, \text{ where } u := \sum_{i=1}^{n-2} u_i, \text{ and } u d\varphi := \sum_{i=1}^{n-2} u_i d\varphi_i.$$

Let $\varepsilon \in \mathbb{R}^+$ be given, consider the contact subdomains of $(\mathbb{R}^{2n-3}, \xi_{\text{st}})$ given by

$$\Delta_{\text{cyl}} = \{z \in [-1, 1 - \varepsilon], u \in [0, 1]\}, \quad \Delta_\varepsilon = \{z \in [-1 + \varepsilon, 1 - \varepsilon], u \in [0, 1 - \varepsilon]\},$$

and define the subset $B = \{z = -1, u \in [0, 1]\} \cup \{z \in [-1, 1 - \varepsilon], u = 1\} \subseteq \partial\Delta_{\text{cyl}}$ of the boundary of Δ_{cyl} . These contact domains are shown in Figure 2.

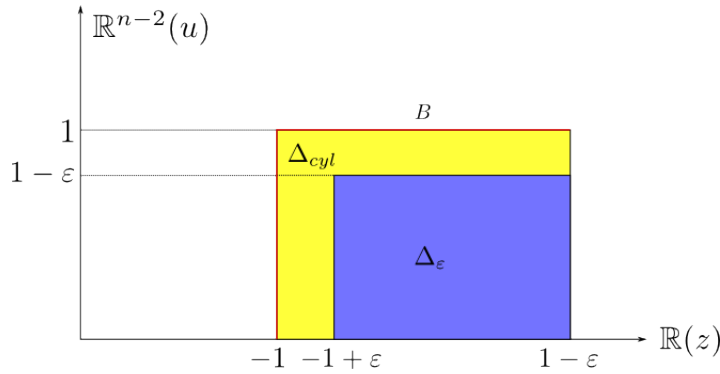


FIGURE 2. The domains Δ_{cyl} in yellow, Δ_ε in blue and B in red. The domains are rotationally symmetric along the z -axis and we implicitly consider the coordinates of the angle φ as included in the graphic representations of these domains.

Let $k_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ be the piecewise linear function defined by

$$k_\varepsilon(x) := \begin{cases} 0 & x \leq 1 - \varepsilon \\ x - (1 - \varepsilon) & x \geq 1 - \varepsilon. \end{cases}$$

and fix a piecewise smooth function $K_\varepsilon : \Delta_{\text{cyl}} \rightarrow \mathbb{R}$ of the form

$$K_\varepsilon(u_i, \varphi_i, z) := \begin{cases} k_\varepsilon(z) + k_\varepsilon(u) & (u_i, \varphi_i, z) \in \Delta_{\text{cyl}} \setminus \text{Int}(\Delta_\varepsilon) \\ < 0 & (u_i, \varphi_i, z) \in \text{Int}(\Delta_\varepsilon). \end{cases}$$

Denote $q = (u_i, \varphi_i, z)$, then the function K_ε defines the following two embeddings of two $(2n - 2)$ -dimensional hypersurfaces:

$$\Sigma_1 = \{(q, v, t) \in \Delta_{\text{cyl}} \times T^*S^1 : t \in S^1, v = K_\varepsilon(q)\} \subseteq (\Delta_{\text{cyl}} \times T^*S^1, \ker(\alpha_0 + vdt))$$

$$\Sigma_2 = \{(q, v, t) \in \Delta_{\text{cyl}} \times \mathbb{C} : q \in B, t \in S^1, v \in [0, K_\varepsilon(q)]\} \subseteq (\Delta_{\text{cyl}} \times \mathbb{C}, \ker(\alpha_0 + vdt)).$$

The coordinates (v, t) represent linear coordinates in T^*S^1 whereas the coordinates (\sqrt{v}, t) represent polar coordinates on \mathbb{C} . Notice that $K_\varepsilon > 0$ on $B \subseteq \partial\Delta_{\text{cyl}}$ and thus Σ_2 is well-defined. Each of the two hypersurfaces Σ_1, Σ_2 defines a germ of a contact structure

$(\mathcal{O}p \Sigma_1, \eta_1), (\mathcal{O}p \Sigma_2, \eta_2)$ inherited from its ambient contact domain. Since $\partial \Sigma_2 \subseteq \partial \Sigma_1$, the union $\Sigma_1 \cup \Sigma_2$ is a piecewise smooth disk. Let us denote this disk, together with the germ of contact structure defined by the embedding, by $(D_{K_\varepsilon}, \eta_{K_\varepsilon})$.

In the article [3], an *overtwisted disk* $(D_{K_{\text{univ}}}, \eta_{K_{\text{univ}}})$ is defined to be a certain contact germ $\eta_{K_{\text{univ}}}$ along a piecewise smooth $(2n-2)$ -disk $D_{K_{\text{univ}}}$, where the definition of K_{univ} is not constructive or canonical. However, we can take $K_{\text{univ}} = K_\varepsilon$ for any sufficiently small $\varepsilon < \varepsilon_{\text{univ}}$, where $\varepsilon_{\text{univ}}$ is a constant depending only on dimension. See Remark 3.2 and Example 3.4 in [3, Section 3]. Considering this, we also denote $(D_{K_\varepsilon}, \eta_{K_\varepsilon})$ by $(D_\varepsilon^{\text{ot}}, \eta_\varepsilon^{\text{ot}})$ if $\varepsilon < \varepsilon_{\text{univ}}$, and refer to this model as an *overtwisted disk*. A contact manifold (Y, ξ) is *overtwisted* if there is a piecewise smooth embedding $D^{2n-2} \subseteq Y$ so that $(D^{2n-2}, \xi|_D^{2n-2})$ is an overtwisted disk.

3.2. The 3-dimensional case. In the definition of an overtwisted disk, we express the disk $(D_\varepsilon^{\text{ot}}, \eta_\varepsilon^{\text{ot}}) = \Sigma_1 \cup \Sigma_2$ as the union of two pieces related to the graph of K_ε . Σ_1 is simply the graph $\{v = K_\varepsilon\} \subseteq \Delta_{\text{cyl}} \times T^*S^1$, and Σ_2 is the sublevel set $\{v \leq K_\varepsilon|_B\} \subseteq \Delta_{\text{cyl}} \times \mathbb{C}$. For Σ_1 , it is essential to have fiber $(v, t) \in T^*S^1$ since K_ε is often negative, but for Σ_2 it is essential that (\sqrt{v}, t) define polar coordinates on \mathbb{C} , in order to be able to define the sublevel set $v \leq K_\varepsilon$ correctly. Much of the work in this section is therefore describing contact structures as being foliated by symplectic surfaces. This is done in such a way that there are enough leaves equivalent to \mathbb{C} or T^*S^1 so that we may embed $\Sigma_1 \cup \Sigma_2$ into our manifold.

First let us describe a local model $(M, \ker \alpha_M)$ which is contained in any overtwisted 3-manifold (Y, ξ) . The domain M is diffeomorphic to an open 3-ball and admits global coordinates (z, v, t) . In these coordinates the contact form globally reads

$$\alpha_M = dz + vdt$$

However, the coordinate $z \in (-3 - \varepsilon, 1)$ dictates the domain of definition of the pair of

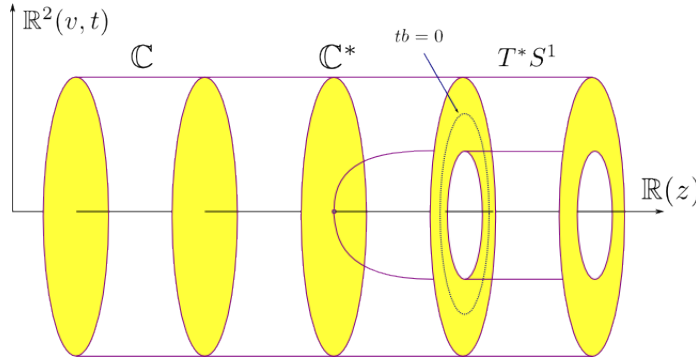


FIGURE 3. The overtwisted contact ball $(M, \ker \alpha_M)$.

coordinates (v, t) . The symplectic submanifolds $\{z = \text{constant}\}$ belong to one of the following three types:

- a. For $z \in (-1 + \frac{2\varepsilon}{3}, 1)$, we have $(v, t) \in (-\infty, \infty) \times S^1$. Thus in this range the submanifolds $\{z = \text{constant}\}$ are exact symplectomorphic to T^*S^1 since the restriction of α equals the canonical Liouville form.
- b. For $z \in [-1 + \frac{\varepsilon}{3}, -1 + \frac{2\varepsilon}{3}]$, we let $t \in S^1$ and $v \in (0, \infty)$. Then the fibers are exact symplectomorphic to $\{v > 0\} \subseteq T^*S^1$. Notice that these fibers are also equal to the standard Liouville structure on $\mathbb{C} \setminus \{0\}$ with polar coordinates (\sqrt{v}, t) .

- c. For $z \in (-3 - \varepsilon, -1 + \frac{\varepsilon}{3})$, we define the fibers $\{z = \text{constant}\}$ to be equal to \mathbb{C} , with (\sqrt{v}, t) continuing to represent polar coordinates.

First, the contact domain $(M, \ker \alpha_M)$ is overtwisted. For example, the Legendrian circle $\{z = \text{constant} > -1 + \frac{2\varepsilon}{3}, v = 0, t \in S^1\}$ is unknotted and has zero Thurston-Bennequin number.

Second, the contact domain $(M, \ker \alpha_M)$ serves as a local model in any overtwisted 3-manifold. Indeed, let (Y, ξ) be an overtwisted 3-manifold and choose an open ball $U \subseteq Y$ which is contained in a compact subset. Let us consider a new contact structure ζ on Y , which is homotopic through plane fields to ξ , equal to ξ outside of a compact subset, and equal to $\ker(\alpha_M)$ on $U \cong M$. This can be arranged by the theorem of R. Lutz and J. Martinet [30, 31]. Since ζ and ξ are overtwisted and equal outside of a compact subset, the uniqueness h -principle [15] implies that they are homotopic with a compactly supported homotopy, and Gray's theorem [26] implies that they are isotopic.

Next, consider the contact manifold $\tilde{\Delta} = (-3 - \varepsilon, 1) \times D^2(1)$, equipped with the standard contact form $dz + u d\varphi$. Here $z \in (-3 - \varepsilon, 1)$ and (\sqrt{u}, φ) are polar coordinates on $D^2(1)$. Let us emphasize that we are not thinking of $\tilde{\Delta}$ as being compared to M . Rather, we consider $\tilde{\Delta}$ as being orthogonal to M , by identifying $M = M \times \{0\} \subseteq M \times D^2(1)$ and $\tilde{\Delta} = \{(v, t) = \text{constant}\} \subseteq M \times D^2(1)$. Taking this into account, we define subdomains

$$\begin{aligned}\tilde{\Delta}_- &= \{z \in (-1 + \frac{2\varepsilon}{3}, 1 - \varepsilon]\} \subseteq \tilde{\Delta} \\ \tilde{\Delta}_+ &= \{z \in [-3, -1 + \frac{\varepsilon}{3}]\} \subseteq \tilde{\Delta}.\end{aligned}$$

Therefore $\tilde{\Delta}_- \subseteq \tilde{\Delta}$ corresponds to those values of (z, u, φ) so that the $(v, t) \in T^*S^1$ for $(z, v, t, u, \varphi) \in M \times D^2(1)$. Similarly, $\tilde{\Delta}_+$ corresponds to those coordinates where the (v, t) fiber is equivalent to \mathbb{C} .

Lemma 3.2. *There exists a contact embedding $f : (\Delta_{\text{cyl}}, \xi_{\text{st}}) \rightarrow (\tilde{\Delta}, \xi_{\text{st}})$ so that $f(\Delta_\varepsilon) \subseteq \tilde{\Delta}_-$ and $f(B) \subseteq \tilde{\Delta}_+$.*

Proof of Theorem 3.1 in the case $n = 2$: Let $\varepsilon \in \mathbb{R}^+$ be such that $\varepsilon < \varepsilon_{\text{univ}}$ and consider the contact embedding $f : \Delta_{\text{cyl}} \rightarrow \tilde{\Delta}$ provided in Lemma 3.2. Consider the positive function

$$c_f : \Delta_{\text{cyl}} \rightarrow (0, \infty), \quad \text{defined by } f^* \alpha_{\text{st}} = c_f \alpha_{\text{st}}$$

and the Hamiltonian $\tilde{K} : f(\Delta_{\text{cyl}}) \rightarrow \mathbb{R}$ given by the equation $\tilde{K} = (c_f \cdot K) \circ f^{-1}$.

We restrict to the local model defined in Section 3.2 and assume that $(Y, \ker \alpha) = (M, \ker \alpha_M)$. If (\sqrt{u}, φ) are polar coordinates on $D^2(1)$, then (z, v, t, u, φ) define coordinates on $M \times D^2(1)$ and this domain is given the contact form $dz + v dt + u d\varphi$.

We provide an overtwisted 4-disk in the 5-dimensional stabilized local model $M \times D^2(1)$. Consider the two hypersurfaces

$$\begin{aligned}\tilde{\Sigma}_1 &= \{(z, v, t, u, \varphi) : (z, u, \varphi) \in f(\Delta_{\text{cyl}}), t \in S^1, v = \tilde{K}(z, u, \varphi)\} \\ \tilde{\Sigma}_2 &= \{(z, v, t, u, \varphi) : f(z, u, \varphi) \in B, t \in S^1, v \in [0, \tilde{K}(z, u, \varphi)]\}.\end{aligned}$$

Notice that $\tilde{\Sigma}_1$ is a well-defined subset of $M \times D^2(1)$ since $\Delta_\varepsilon \subseteq f(\tilde{\Delta}_-)$, and $\tilde{\Sigma}_2$ is also well-defined since $B \subseteq f(\tilde{\Delta}_+)$. Then the contact germ of the 4-disk $D^4 = \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$ defines an overtwisted disk. Indeed, the contactomorphism

$$F(z, u, \varphi, v, t) = (f(z, u, \varphi), v \cdot c_f(z, u, \varphi), t).$$

maps the contact germ $(D_\varepsilon^{\text{ot}}, \eta_\varepsilon^{\text{ot}})$ to $(D^4, \ker(\alpha_M + \lambda_{\text{st}}))$. □

Proof of Lemma 3.2: The contactomorphism f acts by taking the region $\{u \geq 1 - \frac{\varepsilon}{2}\}$ and shearing its z -coordinate far to the left, as depicted in Figure 4. Specifically, the image of f is

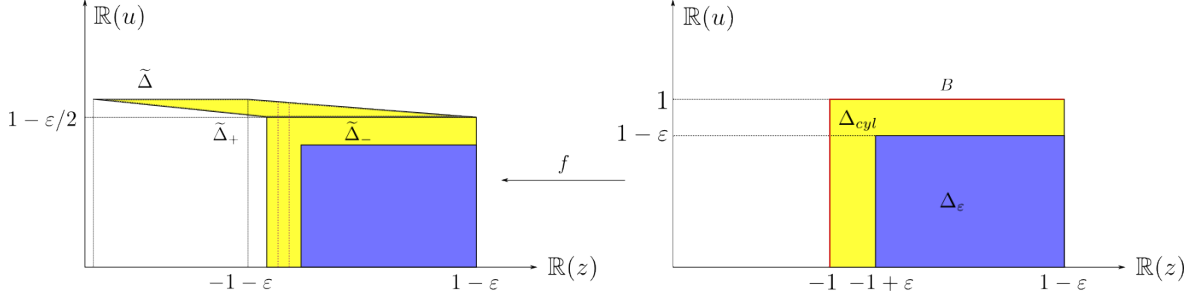


FIGURE 4. The contact domains appearing in Lemma 3.2.

$$\{z \in [-1, 1 - \varepsilon], u \in [0, 1 - \frac{\varepsilon}{2}]\} \cup \{z \in [-3 + \frac{4}{\varepsilon}(1 - u), -1 - \varepsilon + \frac{4}{\varepsilon}(1 - u)], u \in [1 - \frac{\varepsilon}{2}, 1]\} \subseteq \tilde{\Delta}.$$

In the region $\{u \geq 1 - \frac{\varepsilon}{2}\}$ we define the contactomorphism by

$$f(z, u, \varphi) = (z - \frac{4}{\varepsilon}(u - 1 + \frac{\varepsilon}{2}), u, \varphi + \frac{4}{\varepsilon} \log(u)),$$

and we extend it to the region $\{u \leq 1 - \frac{\varepsilon}{2}\}$ by

$$(z, u, \varphi) \mapsto (z, u, \varphi + \frac{4}{\varepsilon} \log(1 - \frac{\varepsilon}{2})).$$

This extension defines the required contactomorphism. \square

3.3. General dimensions. The argument used in order to conclude Theorem 3.1 for an arbitrary overtwisted (Y, ξ) contains the same steps as in the case $n = 2$. However, the definition of the domain $\tilde{\Delta}$ and the local model (M, α_M) are more involved.

For a real number $\rho \gg 0$, We describe a model (M, α_M) using coordinates (z, u, φ, v, t) , where $z \in (-\rho, 1)$, $(u, \varphi) = (u_1, \dots, u_{n-2}, \varphi_1, \dots, \varphi_{n-2})$ are vectorial polar coordinates on the ball $D^{2n-4}(\rho)$. As before, the contact form is globally given by $\alpha_M = dz + u d\varphi + v dt$, but the domain of the variables (v, t) is either T^*S^1 , \mathbb{C}^* or \mathbb{C} depending on the coordinates $(z, u, \varphi) \in (-\rho, 1) \times D^{2n-4}(\rho)$. This dependency is:

- For $(z, |u|) \in (-1 + \frac{2\varepsilon}{3}, 1) \times [0, 1 - \frac{2\varepsilon}{3}]$, we have $(v, t) \in (-\infty, \infty) \times S^1$. In this range, the symplectic submanifolds $\{(z, u, \varphi) = \text{constant}\}$ are exact symplectomorphic to T^*S^1 .
- For $(z, |u|) \in (-1 + \frac{\varepsilon}{3}, -1 + \frac{2\varepsilon}{3}) \times [0, 1 - \frac{\varepsilon}{3}] \cup (-1 + \frac{\varepsilon}{3}, 1) \times (1 - \frac{2\varepsilon}{3}, 1 - \frac{\varepsilon}{3})$, we consider $(v, t) \in (0, \infty) \times S^1$. The symplectic submanifolds $\{(z, u, \varphi) = \text{constant}\}$ are symplectomorphic to \mathbb{C}^* .
- For $\{(z, |u|) \in (-\rho, -1 + \frac{\varepsilon}{3}) \times [0, \rho] \cup (-\rho, 1) \times (1 - \frac{\varepsilon}{3}, \rho)\}$, we have $(v, t) \in \mathbb{C}$ (as polar coordinates (\sqrt{v}, t)).

For any ρ and ε , this local model $(M, \ker \alpha_M)$ exists in every overtwisted contact manifold $(Y, \ker \alpha)$ using the isocontact embedding h-principle [3]. Hence in order to conclude Theorem 3.1 it is left to prove that the contact domain $(M \times D^2(R), \ker(\alpha_M + \lambda_{\text{st}}))$ contains an

overtwisted $2n$ -disk when R is sufficiently large. As in the 3-dimensional case, we denote $\tilde{\Delta} = (-\rho, 1 - \varepsilon) \times D^{2n-4}(\rho) \times D^2(R)$, containing subdomains

$$\begin{aligned}\tilde{\Delta}_- &= \{\text{region a}\} \times D^2(R) \subseteq \tilde{\Delta} \\ \tilde{\Delta}_+ &= \{\text{region c}\} \times D^2(R) \subseteq \tilde{\Delta}.\end{aligned}$$

Just as before, we need to compare the two contact domains $\tilde{\Delta}$ to Δ_{cyl} .

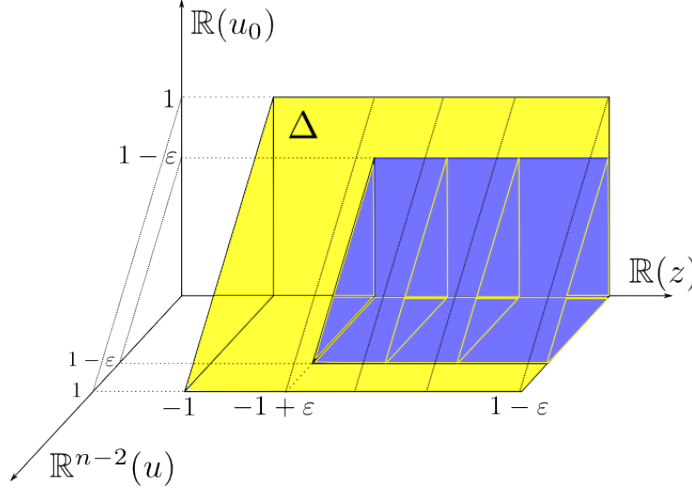


FIGURE 5. The contact domain $\Delta_{\text{cyl}} = (z, u_0, \varphi_0, u, \varphi)$.

Lemma 3.3. *For any $\varepsilon > 0$ there exist $\rho, R > 0$ so that there is a contact embedding $F : \Delta_{\text{cyl}} \rightarrow \tilde{\Delta}$, satisfying $F(\Delta_\varepsilon) \subseteq \tilde{\Delta}_-$ and $F(B) \subseteq \tilde{\Delta}_+$.*

Given the Lemma, the proof of Theorem 3.1 proceeds exactly as it does in the $n = 2$ case. However, the proof of Lemma 3.3 is slightly more involved than the proof of Lemma 3.2.

Proof of Lemma 3.3: Both Δ_{cyl} and $\tilde{\Delta}$ are contact subdomains of $(\mathbb{R}^{2n-1}, \xi_{\text{st}})$, we denote coordinates by $(z, u_0, \varphi_0, u, \varphi)$, where $(\sqrt{u_0}, \varphi_0)$ are coordinates of the $D^2(R)$ factor. Compared to the $n = 2$ case, the new difficulty is that the set $B \subseteq \Delta_{\text{cyl}}$ is larger: with the additional (u, φ) coordinates we have $B = \{z = -1\} \cup \{u_0 + |u| = 1\}$.

Start with the contactomorphism $\tilde{f}(z, u_0, \varphi_0, u, \varphi) = (f(z, u_0, \varphi_0), u, \varphi)$, where f is the contactomorphism defined in Lemma 3.2, then $\tilde{f}(B)$ is not contained in $\tilde{\Delta}_+$. However,

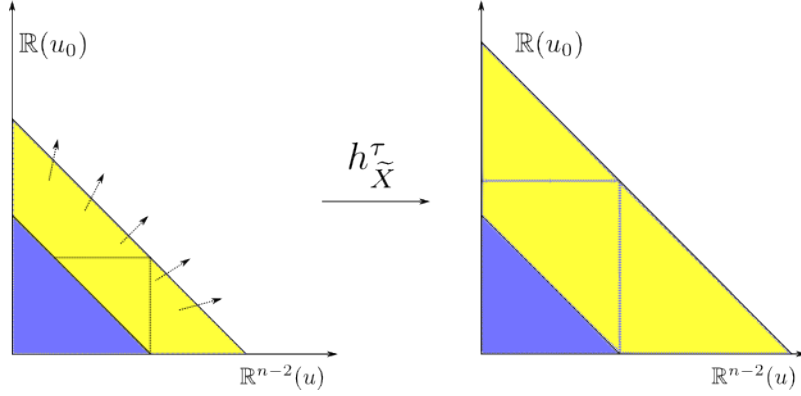
$$\tilde{f}(B) \setminus \Delta_+ \subseteq \{|u| > \frac{\varepsilon^2}{4}\}.$$

In order to see this, suppose that $(z, u_0, \varphi_0, u, \varphi) \in B$ and $|u| \leq \frac{\varepsilon^2}{4}$. Then $u_0 \geq 1 - \frac{\varepsilon^2}{4}$, and by inspection one can verify that the image $f(z, u_0, \varphi_0)$ has z -coordinate less than -1 .

Next, consider the contact vector field $X = u\partial_u + u_0\partial_{u_0} + z\partial_z$ on Δ_{cyl} and cut-off its contact Hamiltonian H to a Hamiltonian \tilde{H} such that its contact vector field \tilde{X} satisfies

- a. \tilde{X} vanishes in $\{z \geq -1 + \frac{2\varepsilon}{3}, u + u_0 \leq 1 - \frac{2\varepsilon}{3}\}$.
- b. \tilde{X} coincides with X in $\{z \leq -1 + \frac{\varepsilon}{3}, 1 - \frac{\varepsilon}{3} \leq u + u_0\}$.

Let $h_{\tilde{X}}^\tau$ be the flow of \tilde{X} , near B $h_{\tilde{X}}^\tau$ acts as radial expansion, see Figure 6. We claim that $F = h_{\tilde{X}}^\tau \circ \tilde{f}$ satisfies the Lemma for sufficiently large τ . Clearly $F(\Delta_\varepsilon) = \Delta_\varepsilon \subseteq \tilde{\Delta}_-$. Note also that \tilde{X} is inwardly transverse to $\partial\tilde{\Delta}_+$ everywhere, so $h_{\tilde{X}}^\tau(\tilde{\Delta}_+) \subseteq \tilde{\Delta}_+$ for any τ . Therefore the only thing to check is that $h_{\tilde{X}}^\tau(\tilde{f}(B) \setminus \tilde{\Delta}_+) \subseteq \tilde{\Delta}_+$. Since the u coordinate on the set

FIGURE 6. Cross section (u, u_0) for the expanded domain $h_X^\tau(\Delta_{\text{cyl}})$.

$\tilde{f}(B) \setminus \tilde{\Delta}_+$ is bounded below by a positive number, and since h_X^τ expands the coordinate u exponentially, we see that for large τ we can arrange $h_X^\tau(\tilde{f}(B) \setminus \tilde{\Delta}_+) \subseteq \{u > 1 - \frac{\varepsilon}{3}\} \subseteq \tilde{\Delta}_+$. \square

4. WEINSTEIN COBORDISM FROM OVERTWISTED TO STANDARD SPHERE

The main goal of this section is proving the equivalence $1 = 3a = 3b$ in Theorem 1.1. Firstly we state an important application of the previous section. The contact branched cover technique [22, 37] along with Theorem 3.1 yield the following class of examples of overtwisted contact structures.

Theorem 4.1. *Let (Y, ξ) be a contact manifold and $(D, \xi|_D)$ a codimension-2 overtwisted contact submanifold. A k -fold contact branched cover of (Y, ξ) along $(D, \xi|_D)$ is overtwisted for k large enough.*

This result follows immediately from Theorem 3.1, since branch covering increases the product neighborhood width of the branch locus. We apply it to prove:

Theorem 4.2. *In every dimension, there is a Weinstein cobordism (W, λ, φ) such that the concave end $(\partial_- W, \lambda)$ is overtwisted and the convex end $(\partial_+ W, \lambda) \cong (S^{2n-1}, \xi_{\text{st}})$.*

Theorem 4.2 is proven assuming the equivalence $1 = 2a$ in Theorem 1.1 which was proven in Section 3, it also uses the inductive hypothesis in the dimension n . The Weinstein cobordism (W, λ, φ) is smoothly non-trivial and it is constructed such that $\partial_- W$ is a sphere.

Proof. We construct a Weinstein cobordism $(W^{2n}, \lambda, \varphi)$ of finite type from an overtwisted contact structure $(S^{2n-1}, \xi_{\text{ot}})$ to the standard contact sphere $(S^{2n-1}, \xi_{\text{st}})$. The construction has two steps.

Let A_k^{2n-2} be the Milnor fibre obtained as an A_k -plumbing of k copies of T^*S^{n-1} , with its induced Weinstein structure. First, we prove that the contact manifold

$$(S^{2n-1}, \xi_k) = \text{OB}(A_{2k-1}^{2n-2}, \tau_1^{-1} \circ \dots \circ \tau_{2k-1}^{-1})$$

is overtwisted for k large enough.

Using the inductive hypothesis specifically $1 = 5$ in Theorem 1.1, we can assume that $(S^{2n-3}, \xi_1) = \text{OB}(T^*S^{n-2}, \tau^{-1})$ is overtwisted. It also admits a contact embedding into $(S^{2n-1}, \xi_1) = \text{OB}(A_1^{2n-2}, \tau^{-1})$ compatible with the open book decomposition which corresponds to the cotangent bundle of an unknotted equatorial $S^{n-2} \subseteq S^{n-1}$. Then Theorem 4.1 implies that the k -branched cover (Y_k, ζ_k) of (S^{2n-1}, ξ_1) along (S^{2n-3}, ξ_1) is an overtwisted contact manifold. Note that Y_k is diffeomorphic to S^{2n-1} because S^{2n-3} is a smooth unknot.

We show that the contact structure (S^{2n-1}, ζ_k) is supported by the open book decomposition $\text{OB}(A_{2k-1}^{2n-2}, \tau_1^{-1} \circ \dots \circ \tau_{2k-1}^{-1})$ and hence it is contact isotopic to (S^{2n-1}, ξ_k) , which concludes the assertion. First note that the projection map for the open book $\text{OB}(A_1^{2n-2}, \tau_1^{-1})$ is given by argument of the map

$$f : S^{2n-1} \subset \mathbb{C}^{2n} \longrightarrow \mathbb{C}, \quad f(z_1, \dots, z_n) = \bar{z}_1^2 + \dots + \bar{z}_n^2.$$

Then the overtwisted submanifold (S^{2n-3}, ξ_1) is cut out by the equation $\{z_1 = 0\}$ and the k -branched cover along it can be realized by the map $z_1 \mapsto z_1^k$. Thus the contact structure (Y_k, ζ_k) is supported by the open book induced by the argument of the map

$$f : S^{2n-1} \subset \mathbb{C}^{2n} \longrightarrow \mathbb{C}, \quad f(z_1, \dots, z_n) = \bar{z}_1^{2k} + \bar{z}_2^2 + \dots + \bar{z}_n^2,$$

which is $\text{OB}(A_{2k-1}^{2n-2}, \tau_1^{-1} \circ \dots \circ \tau_{2k-1}^{-1})$.

Finally, we show that (S^{2n-1}, ξ_k) is Weinstein cobordant to $(S^{2n-1}, \xi_{\text{st}})$. Notice that

$$\text{OB}(A_{2k-1}^{2n-2}, \tau_1 \circ \dots \circ \tau_{2k-1}) = (S^{2n-1}, \xi_{\text{st}})$$

since this open book is just the trivial open book (D^{2n-2}, id) positively stabilized $2k-1$ times. But then by doing two Weinstein handle attachments (as in Proposition 2.10) to each zero section in A_{2k-1}^{2n-2} , for a total of $4k-2$ handles, we form a cobordism from (S^{2n-1}, ξ_k) to $(S^{2n-1}, \xi_{\text{st}})$. \square

Proposition 4.3. *Suppose that the standard unknot is loose in (Y, ξ) . Let (W, λ, φ) be any Weinstein cobordism and let SY be the symplectization of Y . Then $W \# SY$ is flexible.*

Proof. Since SY is a trivial product, the critical points of $W \# SY$ are the same as the critical points of W . Let p be a critical point in $W \# SY$ of index n , let $M = \varphi^{-1}(\varphi(c) - \varepsilon)$ be a level set of W , and let $\Lambda \subseteq M \# Y$ be the Legendrian attaching map of p . Let $\Lambda_0 \subseteq Y$ be the standard Legendrian unknot, and let $U \subseteq M \# Y$ be the union of a Darboux chart containing Λ_0 and a loose chart for Λ_0 . Since Λ_0 is loose as a Legendrian in Y and since Λ is the descending manifold of a critical point of W , we know that Λ is disjoint from U , and therefore $\Lambda \# \Lambda_0$ is a loose Legendrian (with the same loose chart as Λ_0). But $\Lambda \# \Lambda_0$ is contact isotopic to Λ , since Λ_0 is the standard unknot. It follows that Λ is loose. \square

Theorem 4.4. *Let Λ_0 be the standard Legendrian unknot inside a contact manifold (Y, ξ) . If Λ_0 is a loose Legendrian then (Y, ξ) is overtwisted.*

Proof. Let W be the cobordism constructed in Theorem 4.2. Then Proposition 4.3 tells us that $W \# SY$ is flexible. $\partial_-(W \# SY) = \partial_- W \# Y$ is overtwisted since $\partial_- W$ is, so 2.8 implies that $\partial_+(W \# SY) = (S^{2n-1}, \xi_{\text{st}}) \# (Y, \xi) \cong (Y, \xi)$ is overtwisted as well. \square

The standard unknot in (Y, ξ) is defined by the inclusion of a small Darboux chart in Y . If (Y, ξ) contains a small plastikstufe with trivial rotation, the unknot must be in the complement. Therefore, Theorems 2.5 and 4.4 imply

Theorem 4.5. *Let (Y, ξ) be a contact manifold containing a small plastikstufe with spherical core and trivial rotation. Then (Y, ξ) is overtwisted.*

5. (+1)-SURGERY ON LOOSE LEGENDRIANS

In this section we prove the equivalence 1 = 4 in Theorem 1.1 by using Theorem 4.4. For our purpose, we use the following model of contact (+1)-surgery on a Legendrian sphere, defined in the article [1, Section 9]:

Definition 5.1. Let $\Lambda \subseteq (Y, \xi)$ be a Legendrian sphere in a contact manifold. A neighborhood of Λ can be identified with (a neighborhood of the zero section in) the first-jet space $\mathcal{J}^1(S^{n-1}) = T^*S^{n-1} \times \mathbb{R}$, and let $\tau : T^*S^{n-1} \rightarrow T^*S^{n-1}$ be the Dehn–Seidel twist. Remove $D^*S^{n-1} \times (0, 1)$ from Y , and then glue the boundary to itself with the identification $(x, 0) \sim (\tau^{-1}(x), 1)$ and $(x, t) \sim (x, t')$ for $x \in \partial D^*S^{n-1}$. This is a smooth manifold Y' since the diffeomorphism τ is compactly supported, and it has a canonical contact structure ξ' since τ is a symplectomorphism. We call the contact manifold (Y', ξ') the *contact (+1)–surgery of (Y, ξ) along Λ* .

Remark 5.2. Given a Legendrian sphere $\Lambda \subseteq (Y, \xi)$, the contactomorphism type of (Y', ξ') depends on a parametrization $f : S^{n-1} \rightarrow \Lambda$. In fact [14, Theorem A] shows that the class $[\tau] \in \pi_0 \text{Sym}(T^*S^{n-1})$ genuinely depends on this parametrization. However, in our context we can completely ignore this distinction since any two parametrizations of *loose* Legendrian spheres are ambiently contact isotopic [33, Theorem 1.2].

In this surgery model, we can prove the equivalence $3b = 4$ in Theorem 1.1.

Theorem 5.3. *Let $\Lambda \subseteq (Y, \xi)$ be a loose Legendrian submanifold. Then the contact (+1)–surgery of (Y, ξ) along Λ contains a small plastikstufe with spherical core and trivial rotation.*

Proof. Since the Legendrian sphere Λ is loose, we can choose a Legendrian sphere $\tilde{\Lambda}$ such that spherically stabilizing $\tilde{\Lambda}$ gives the Legendrian Λ . Choose coordinates in a neighborhood of the Legendrian $\tilde{\Lambda}$ identifying it with a neighborhood of the zero section in the jet space $T^*S^{n-1} \times \mathbb{R}$. We can then represent the original Legendrian Λ as the zero section stabilized over the equator $S^{n-2} \subseteq S^{n-1}$. For a fixed point $x \in S^{n-2}$, consider the unique meridian $S_x^1 \subseteq S^{n-1}$ passing through the point x and the submanifold $\mathcal{J}^1(S_x^1) \subseteq T^*S^{n-1} \times \mathbb{R}$. The jet space $\mathcal{J}^1(S_x^1)$ is a 3-dimensional contact submanifold contactomorphic to $T^*S^1 \times \mathbb{R}$, and under this contactomorphism the intersection $\Lambda \cap \mathcal{J}^1(S_x^1)$ is given as the stabilization of the zero section. Note also that for $x \neq y$, we can identify $\mathcal{J}^1(S_x^1) \cap \mathcal{J}^1(S_y^1) \cong S^0 \times \mathbb{R}$ where S^0 is the union of the north and south poles.

Because the symplectomorphism τ is defined using the geodesic flow on the sphere and the meridian S_x^1 is a geodesic submanifold, we see that the Dehn twist τ preserves the submanifold $T^*S_x^1$. Letting $q : (Y \setminus \mathcal{O}p(\Lambda), \xi) \rightarrow (Y', \xi')$ be the quotient map realizing the contact (+1)–surgery on Λ , we see that $q(\mathcal{J}^1(S_x^1))$ is a contact submanifold M_x which is itself contactomorphic to the contact (+1)–surgery of $\mathcal{J}^1(S_x^1)$ along the stabilized Legendrian $\Lambda \cap \mathcal{J}^1(S_x^1)$. In particular, the manifold (M_x, ξ') is overtwisted for every $x \in S^{n-2}$, even in the complement of the submanifold $S^0 \times \mathbb{R}$. The entire picture is symmetric about $x \in S^{n-2}$, thus the construction defines a plastikstufe P with spherical core.

It remains to show that P has trivial rotation class and that it is contained in a smooth ball. We prove these claims simultaneously, by showing that an open leaf of P is contained in a Legendrian disk. An open leaf of P is given as the union of Legendrian arcs in M_x . Consider an isotopy between this arc and a small Legendrian arc in $\Lambda \cap \mathcal{J}^1(S_x^1)$ disjoint from $S^0 \times \mathbb{R}$. Considering this symmetrically with respect to $x \in S^{n-2}$, we get an isotopy from an open leaf of P to an annulus $S^{n-2} \times [0, 1] \subseteq \Lambda$, and since Λ is a sphere this annulus extends to a disk in Λ . \square

6. STABILIZATION OF LEGENDRIANS AND OPEN BOOKS

In this section we prove the equivalence $3a = 5$ in Theorem 1.1. To do this, the section relates two known procedures in contact topology: the stabilization of a Legendrian submanifold and the stabilizations of a compatible open book ¹. This is explained in Subsection 6.3. The

¹The operation on Legendrians has no intrinsic relationship to the operation on open books. Unfortunately, this confusing terminology has been around long enough to have become quite stable.

link between these two procedures can be established through Lagrangian surgery [35], also referred to as Polterovich surgery. The details regarding Lagrangian surgery are detailed in Subsection 6.2.

The results in Subsections 6.2 and 6.3 imply the following result.

Theorem 6.1. *Let (S^{2n-1}, ξ_-) be the contact manifold supported by the open book whose page is $(T^*S^{n-1}, \lambda_{st})$ and whose monodromy is the left handed Dehn twist along the zero section. Then the standard Legendrian unknot in (S^{2n-1}, ξ_-) is loose.*

In light of Theorem 2.11, this Theorem implies $5 \Rightarrow 3a$. The fact that any overtwisted contact manifold admits a negatively stabilized open book follows quickly from known results. Let (Y, ξ) be an overtwisted contact structure. Since the set of almost contact structures on the sphere forms a group, the existence theorem from [3] implies that there is an overtwisted contact structure (Y, η) so that $(Y \# S^{2n-1}, \eta \# \xi_-)$ is in the same homotopy class of almost contact structures as (Y, ξ) . Since the contact structures ξ and $\eta \# \xi_-$ are both overtwisted, they are necessarily isotopic. By Giroux's existence theorem for open books [24] (given here as Theorem 2.9), the contact structure (Y, η) is compatible with an open book (W, φ) . The negative stabilization of the open book (W, φ) supports the contact structure $(Y, \eta \# \xi_-)$, which is isotopic to (Y, ξ) . This shows the implication $1 \Rightarrow 5$, therefore Theorem 6.1 is the remaining ingredient for the equivalence $1 = 5$.

6.1. Legendrians in open books. In order to prove Theorem 6.1, we develop some combinatorics for describing Legendrian submanifolds in adapted open books decompositions. Let $(Y, \xi) = \text{OB}(W, \lambda, \varphi)$, and recall that if $L \subseteq (W, \lambda)$ is an exact Lagrangian, it determines a Legendrian $\Lambda \subseteq Y$ (see Section 2.4). Denoting this relationship by $(Y, \xi, \Lambda) = \text{OB}(W, \lambda, \varphi, L)$, note that $\text{OB}(W, \lambda, \varphi, L)$ is isotopic to the Legendrian defined by $\text{OB}(W, \lambda, \psi \circ \varphi \circ \psi^{-1}, \psi(L))$, and typically distinct from the Legendrian defined by $\text{OB}(W, \lambda, \psi \circ \varphi \circ \psi^{-1}, L)$. Also the Legendrian $\text{OB}(W, \lambda, \varphi, L)$ is isotopic to $\text{OB}(W, \lambda, \varphi, \varphi(L))$, since the Reeb flow from time 0 to 2π gives a Legendrian isotopy between them. These observations are relevant to the proof and understanding of Theorem 6.1.

The next subsection contains the results expressing Lagrangian surgery on two Lagrangians in terms of the Legendrian connected sum of their Legendrian lifts.

6.2. Lagrangian Surgery and Legendrian Sums. Compactly supported exact symplectomorphisms of a Liouville domain are often times given as compositions of Dehn–Seidel twists [39][Chapter I.2]. It is thus relevant to reinterpret the action of Dehn twists on Lagrangians in terms of their Legendrian lifts. This is the aim of this subsection.

We focus on the case where $L \subseteq (W, \lambda)$ is an exact Lagrangian and $S \subseteq W$ is a Lagrangian sphere intersecting L in one point. In this case, the Dehn twist of L around S can be interpreted as the the Polterovich surgery [20, 35] of L and S , denoted by $L + S$:

Theorem 6.2 ([40]). *The Lagrangian surgery $S + L$ is Lagrangian isotopic to $\tau_S(L)$. The Lagrangian surgery $L + S$ is Lagrangian isotopic to $\tau_S^{-1}(L)$.*

We now model this operation in terms of the fronts of Legendrian lifts Λ and Σ of the Lagrangians L and S , see Figure 7 for a pictorial description of the forthcoming result. The conclusion can be stated as follows:

Theorem 6.3. *The Legendrian lift of $S + L$ is isotopic to the Legendrian cusp sum of Λ and Σ . The Legendrian lift of $L + S$ is isotopic to the Legendrian cone sum of Λ and Σ .*

The rest of the subsection proves Theorem 6.3.

Consider local coordinates $(q_1, \dots, q_{n-1}, p_1, \dots, p_{n-1}) \in \mathbb{R}^{2n-2}$ such that the Lagrangians L and S are locally expressed as $L = \{p_1 = 0, \dots, p_{n-1} = 0\}$, $S = \{q_1 = p_1, \dots, q_{n-1} = p_{n-1}\}$

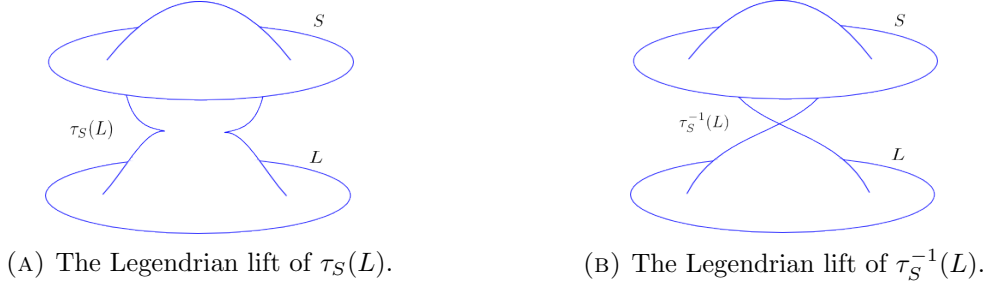


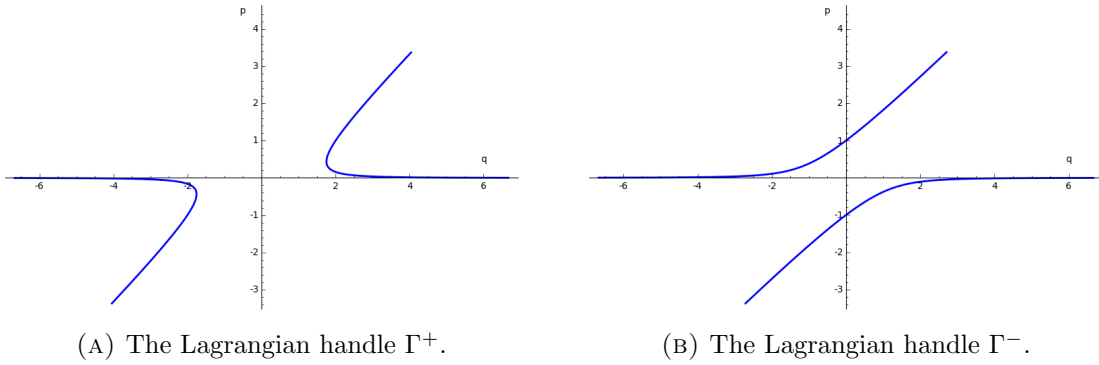
FIGURE 7. The statement of Theorem 6.3.

and the Liouville form reads

$$\lambda_{\text{st}} = \sum_{i=1}^{n-1} p_i dq_i.$$

In the contactization $(\mathbb{R}^{2n-1}(q, p; z), \ker(dz - \lambda_{\text{st}}))$ of the exact (Weinstein) symplectic manifold $(\mathbb{R}^{2n-2}(q, p), \lambda_{\text{st}})$, the Lagrangian L lifts to the Legendrian $\Lambda = \{(q_1, \dots, q_{n-1}, 0, \dots, 0; 0)\}$ and the Lagrangian S lifts to the Legendrian $\Sigma = \{(q_1, \dots, q_{n-1}, q_1, \dots, q_{n-1}; (q_1^2 + \dots + q_{n-1}^2)/2)\}$.

The Lagrangian surgeries $S + L$ and $L + S$ are respectively described in terms of two Lagrangian handles Γ^\pm [35]. These Lagrangian handles are depicted in Figure 8. In order to parametrize them we use coordinates $t = (t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1}$.

FIGURE 8. The Lagrangian handles $\Gamma^\pm \subseteq \mathbb{R}^{2n-2}(q, p)$.

First, we consider the case of the Lagrangian handle Γ^+ . Let us describe it via the parametrization $\Gamma^+ : \mathbb{R}^{n-1} \setminus \{0\} \rightarrow \mathbb{R}^{2n-2}$ defined as

$$\Gamma^+(t_1, \dots, t_{n-1}) = ((\mu + \mu^{-1})t_1, \dots, (\mu + \mu^{-1})t_{n-1}, \mu t_1, \dots, \mu t_{n-1}) \text{ where } \mu = \sum_{i=1}^{n-1} t_i^2.$$

Note that we have $\lim_{\mu \rightarrow \infty} \Gamma^+ \subseteq S$ and $\lim_{\mu \rightarrow 0} \Gamma^+ \subseteq L$. We can lift the exact Lagrangian Γ^+ to the contactization via $z = z(t_1, \dots, t_{n-1})$:

$$\begin{aligned} dz(t) &= \sum_{i=1}^{n-1} \mu t_i d((\mu + \mu^{-1})t_i) = \sum_{i=1}^{n-1} (\mu^2 + 1)t_i dt_i + \sum_{i=1}^{n-1} \mu t_i^2 (1 - \mu^{-2})d\mu = \\ &= \sum_{i=1}^{n-1} (\mu^2 + 1)t_i dt_i + (\mu^2 - 1)d\mu \end{aligned}$$

Hence the partial derivatives of $z(t)$ are:

$$\partial_i z(t) = (\mu^2 + 1)t_i dt_i + (\mu^2 - 1)2t_i dt_i = (3\mu^2 - 1)t_i dt_i.$$

Thus the z -coordinate of the lift is parametrized by $z(t) = \frac{1}{2}(\mu^3 - \mu)$ and in the front projection $\mathbb{R}^n(q_1, \dots, q_{n-1}, z)$ we obtain a rotationally symmetric cusp. Part of the front projections in dimensions 3 and 5 are depicted in Figures 9 and 10. This describes the Polterovich surgery $S + L$ in terms of the cusp-sum of the two Legendrians Λ and Σ respectively lifting L and S , and concludes the first statement of Theorem 6.3.

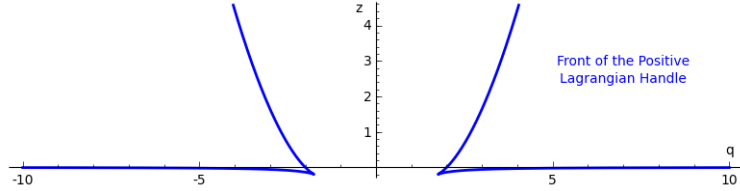


FIGURE 9. Front projection to $\mathbb{R}^2(q_1, z)$ of the Legendrian lift of the positive Lagrangian handle $\Gamma^+ \subseteq \mathbb{R}^3(q_1, p_1, z)$ for $t \in [-1.5, -0.1] \cup [0.1, 1.5]$.

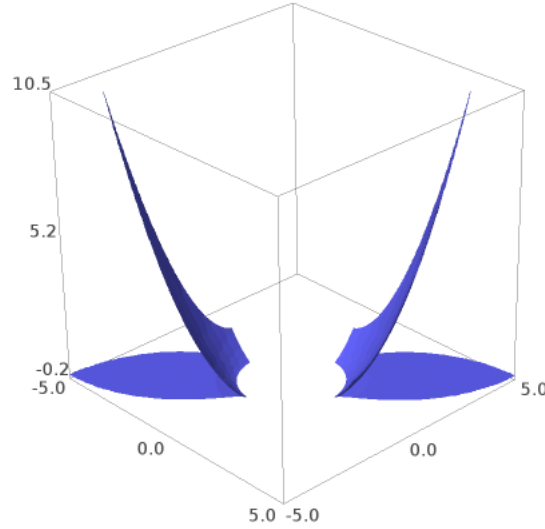


FIGURE 10. Front projection to $\mathbb{R}^3(q_1, q_2, z)$ of the Legendrian lift of $\Gamma^+ \subseteq \mathbb{R}^5$ with (t_1, t_2) in the range $[-1.2, -0.1] \times [-1.2, -0.1] \cup [0.1, 1.2] \times [0.1, 1.2]$.

The Polterovich surgery $L + S$ is described in terms of the Lagrangian handle Γ^- , which yields the cone-sum. Indeed, consider the parametrization of the handle

$$\Gamma^- : \mathbb{R}^{n-1} \setminus \{0\} \longrightarrow \mathbb{R}^{2n-2}, \quad \Gamma^-(t_1, \dots, t_{n-1}) = ((\mu - \mu^{-1})t_1, \dots, (\mu - \mu^{-1})t_{n-1}, \mu t_1, \dots, \mu t_{n-1}).$$

We also have $\lim_{\mu \rightarrow \infty} \Gamma^- \subseteq S$ and $\lim_{\mu \rightarrow 0} \Gamma^- \subseteq L$, and the z -coordinate of the lift to the contactization satisfies

$$dz(t) = \sum_{i=1}^{n-1} \mu t_i d((\mu - \mu^{-1})t_i) = \sum_{i=1}^{n-1} (\mu^2 - 1)t_i dt_i + (\mu^2 + 1)d\mu.$$

We conclude that the partial derivatives of $z(t)$ are $\partial_i z(t) = (3\mu^2 + 1)t_i dt_i$ and $z(t) = \frac{1}{2}(\mu^3 + \mu)$ provides a lift for Γ^- . These front projections are depicted in Figures 11 and 12, and this concludes the second statement of Theorem 6.3.

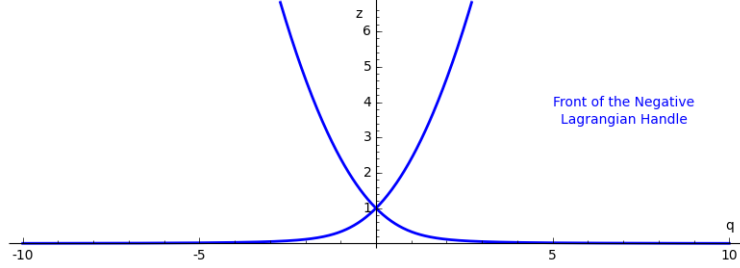


FIGURE 11. Front projection to $\mathbb{R}^2(q_1, z)$ of the Legendrian lift of the handle $\Gamma^- \subseteq \mathbb{R}^3(q_1, p_1, z)$ with $t \in [-1.5, -0.1] \cup [0.1, 1.5]$.

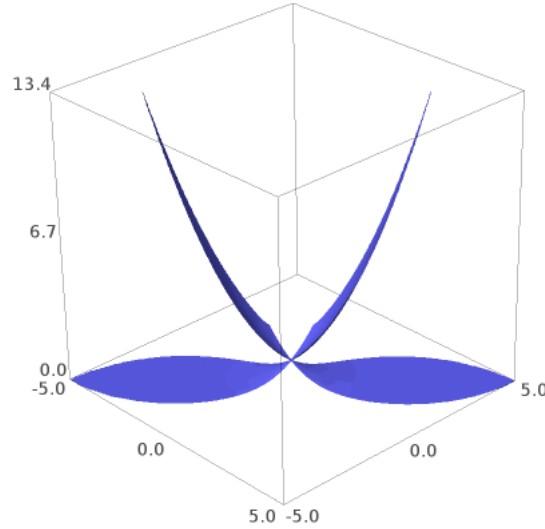


FIGURE 12. Front projection to $\mathbb{R}^3(q_1, q_2, z)$ of the Legendrian lift of $\Gamma^- \subseteq \mathbb{R}^5$ with parameters $(t_1, t_2) \in [-1.2, -0.1] \times [-1.2, -0.1] \cup [0.1, 1.2] \times [0.1, 1.2]$.

6.3. Loose Legendrians in open books. In order to show that the Legendrian unknot in the contact manifold $(S^{2n-1}, \xi_-) = \text{OB}(T^*S^{n-1}, \tau^{-1})$ is loose, we need an understanding of looseness and the standard unknot in the open book framework. This is the content of Propositions 6.4 and 6.5.

Proposition 6.4. *Let $(Y, \xi) = \text{OB}(W, \lambda, \varphi)$ be a contact manifold and $(W \cup H, \lambda, \varphi \circ \tau_S)$ a positive stabilization. The Legendrian lift of S to (Y, ξ) is the standard unknot.*

Proof. First, we note that positive stabilization of an open book can be thought of as connect summing (Y, ξ) with $(S^{2n-1}, \xi_{\text{st}}) = \text{OB}(T^*S^{n-1}, \lambda_{\text{st}}, \tau_S)$, where S denotes the zero section. Therefore it suffices to show that the Legendrian lift of S in this one model is the standard unknot. For this, notice that this open book can be thought of as the boundary of the Lefschetz fibration $f : \mathbb{C}^n \rightarrow \mathbb{C}$ given by $f(z_1, \dots, z_n) = z_1^2 + \dots + z_n^2$. Then S is Hamiltonian isotopic to a vanishing cycle of the unique critical point at 0. Since $\lambda_{\text{st}}|_S = 0$, the Legendrian lift of S is simply the inclusion of S into a single page of the open book, meaning that f is constant on S .

Taking any path $\gamma : [0, 1] \rightarrow \mathbb{C}$ satisfying $\gamma(0) = 0$ and $\gamma(1) = f(S)$, we can use symplectic parallel transport to find a Lagrangian disk $L_\gamma \subseteq \mathbb{C}^n$ so that $\partial L_\gamma = S$. One definition of the standard Legendrian unknot is the Legendrian submanifold which is the boundary of the standard Lagrangian plane $\Lambda_0 = \{y_i = 0\} \cap S_{\text{st}}^{2n-1} \subseteq \mathbb{C}^n$. Therefore, it suffices to show that L_γ is Hamiltonian isotopic to the flat Lagrangian plane.

Since $L_\gamma \cap f^{-1}(\gamma(t))$ is Hamiltonian isotopic to the zero section S , we can choose coordinates so that $\lambda|_{L_\gamma} = 0$. This means that the flow of the Liouville vector field, which is a conformal symplectomorphism, takes L_γ to a submanifold which is C^1 -close to its linearization at $0 \in \mathbb{C}^n$. Using Moser's theorem, we see that L_γ is Hamiltonian isotopic to a flat plane. \square

Proposition 6.5. *Let $(W \cup H, \lambda, \varphi \circ \tau_S)$ be a positively stabilized open book and $L \subseteq W \cup H$ be an exact Lagrangian which transversely intersects S in one point. Then the Legendrian $(W \cup H, \lambda, \varphi \circ \tau_S, L)$ is isotopic to the Legendrian $(W \cup H, \lambda, \varphi \circ \tau_S, \tau_S^{-1}(L))$ and the Legendrian $(W \cup H, \lambda, \varphi \circ \tau_S, \tau_S(L))$ is loose.*

Proof. Choose a Legendrian lift for L which has $\theta = 0$ at $L \cap S$, and a Legendrian lift for S which is just $\theta = \varepsilon$ for some small constant $\varepsilon > 0$ (this is a Legendrian lift since $\lambda|_S = 0$). Theorem 6.3 implies that the Legendrian lifts of $\tau_S(L)$ and $\tau_S^{-1}(L)$ are respectively the cusp and cone sums of the corresponding Lagrangians. Indeed, since they intersect in one point, we know by Theorem 6.2 that $\tau_S(L) = S + L$ and $\tau_S^{-1}L = L + S$. Then by Theorem 6.3 Legendrian lift of $S + L$ corresponds to the cusp-sum, and the Legendrian lift of $L + S$ corresponds to their cone-sum. Since the Legendrian lift of S is the Legendrian unknot, it is contained in a Darboux ball which is disjoint from Λ , and since any two Darboux balls are contact isotopic we see that cone or cusp summing with the unknot is a local operation of Λ .

For the Legendrian $\text{OB}(W \cup H, \lambda, \varphi \circ \tau_S, \tau_S^{-1}(L))$, we note that cone-summing a Legendrian with a small Legendrian unknot does not change the Legendrian isotopy type since this is just the S^{n-2} -spinning of the first Legendrian Reidemeister move. Therefore Λ is isotopic to the lift of $L + S = \tau_S^{-1}(L)$.

For the Legendrian $\text{OB}(W \cup H, \lambda, \varphi \circ \tau_S, \tau_S(L))$, observe that the cusp-sum of a Legendrian with a small Legendrian unknot explicitly creates a loose chart [11, 33] and therefore $\tau_S(L) = S + L$ is loose. \square

6.4. Proof of Theorem 6.1. Consider the contact manifold $(S^{2n-1}, \xi_-) = \text{OB}(T^*S^{n-1}, \tau_L^{-1})$ and stabilize the open book using a cotangent fiber. The Weinstein page $(W, \lambda) = T^*S^{n-1} \cup H$ of the resulting open book is a plumbing of two copies of T^*S^{n-1} whose zero sections L and S intersect in one point.

The Legendrian $\Lambda_\ell = \text{OB}(W, \lambda, \tau_L^{-1} \circ \tau_S, \tau_S(L))$ is loose by Lemma 6.5 and the Legendrian $\Lambda_0 = \text{OB}(W, \lambda, \tau_L^{-1} \circ \tau_S, S)$ is the standard unknot by Proposition 6.4. It suffices to show that they are isotopic. Λ_0 is isotopic to $\text{OB}(W, \lambda, \tau_L^{-1} \circ \tau_S, (\tau_L^{-1} \circ \tau_S)(S))$ because the monodromy is a contactomorphism, and $(\tau_L^{-1} \circ \tau_S)(S) = \tau_L^{-1}(S) = S + L = \tau_S(L)$. \square

7. SOME APPLICATIONS

In this section we explore possible consequences of Theorem 1.1.

7.1. Neighborhood size and contact squeezing. Theorem 1.1 emphasizes in its first equivalence the importance of the size of a neighborhood of a contact submanifold. In this direction it is relevant to understand the dichotomy between tight and overtwisted contact structures in terms of small and large neighborhoods.

Theorem 7.1. *Let $(Y, \ker \alpha)$ be an overtwisted contact manifold. There exists a radius $R_0 > 0$ such that for any $R > R_0$, there exists a compactly supported contact isotopy f_t of $(Y \times \mathbb{C}, \ker\{\alpha + \lambda_{st}\})$ with $f_1(Y \times D^2(R)) \subseteq Y \times D^2(R_0)$.*

This follows immediately from the $1 = 2a$ equivalence in Theorem 1.1 together with the h -principle for isocontact embeddings into overtwisted manifolds [3]. Theorem 7.1, being a contact squeezing result, relates to (weak) non-orderability [3, 17, 23]. The radius R_0 in the statement of Theorem 7.1 can be taken to be twice the minimal radius R_c such that the contact manifold $(Y \times D^2(R_c), \ker\{\alpha + \lambda_{st}\})$ is overtwisted. In contrast with Theorem 7.1, there are instances of contact non-squeezing:

Proposition 7.2. *Let $(Y, \ker \alpha)$ be a contact 3-manifold. Then there exists $\delta \in \mathbb{R}^+$ such that for any $R > \delta$ there is no contact embedding*

$$(Y \times D^2(R), \ker\{\alpha + \lambda_{st}\}) \longrightarrow (Y \times D^2(\delta), \ker\{\alpha + \lambda_{st}\}).$$

This proposition follows from [10, Proposition 11] and known obstructions to fillability [37].

Remark 7.3. Proposition 7.2 also holds for any weakly fillable $(Y^{2n-1}, \ker \alpha)$ [4].

We also observe that the equivalence $1 = 2b$ shows that contactomorphism type is sensitive to dimensional stabilization.

Corollary 7.4. *There exist smooth manifolds Y with two non-isomorphic contact structures $\ker \alpha_1$ and $\ker \alpha_2$ such that $(Y \times \mathbb{C}, \ker\{\alpha_1 + \lambda_{st}\})$ and $(Y \times \mathbb{C}, \ker\{\alpha_2 + \lambda_{st}\})$ are contactomorphic.*

This is an exercise in algebraic topology. For instance, we can consider $\ker \alpha_1$ and $\ker \alpha_2$ to be two different overtwisted contact structures on any homology 3-sphere. Then the two hyperplane fields $\ker\{\alpha_1 + \lambda_{st}\}$ and $\ker\{\alpha_2 + \lambda_{st}\}$ become homotopic as almost contact structures in $Y \times \mathbb{C}$.

This result could be contrasted with [13]. Notice the homotopy class of a compatible almost complex structure distinguishes the symplectizations of two different overtwisted contact structures on S^3 . In particular they are not symplectomorphic. Hence Theorem 7.4 shows that the contactizations of two non-isomorphic exact symplectic structures can be contactomorphic.

7.2. Weinstein cobordisms with overtwisted concave end. In the case $n \geq 3$, a smooth concordance between an overtwisted and a tight contact structure on the sphere S^{2n-1} can be constructed as follows.

Proposition 7.5. *Suppose that $n \geq 3$, then there is a Weinstein structure (M, λ, f) on the smoothly trivial cobordism $M \cong [0, 1] \times S^{2n-1}$ such that $(\partial_+ M, \lambda) \cong (S^{2n-1}, \xi_{st})$ and $(\partial_- M, \ker(\lambda))$ is the unique overtwisted contact sphere in the same almost contact class as ξ_{st} .*

Proof. Let (S^{2n-1}, ξ_{ot}) be the overtwisted contact sphere in the standard almost contact class and let M be its symplectization. The standard Weinstein structure on M can be homotoped to one with a cancelling pair of critical points, one of index $n-1$ and one of index n . Consider a middle contact level (Y, ξ) between these two critical points. Then we can view Y either as a subcritical isotropic surgery on (S^{2n-1}, ξ_{ot}) induced by the bottom half of the cobordism M , or as the result of a $(+1)$ -surgery along a Legendrian sphere $\Lambda \subseteq (S^{2n-1}, \xi_{ot})$ induced by the top half of the cobordism M . The contact structure (Y, ξ) can be obtained as a subcritical surgery on an overtwisted manifold and thus it is overtwisted. See Figure 13 for a schematic picture of the forthcoming argument.

Now let $\Lambda_0 \subseteq (S^{2n-1}, \xi_{st})$ be the loose Legendrian sphere which is in the same formal Legendrian isotopy class as the Legendrian sphere $\Lambda \subseteq (S^{2n-1}, \xi_{ot})$. Note that (S^{2n-1}, ξ_{ot}) and (S^{2n-1}, ξ_{st}) are in the same almost contact class, hence the formal Legendrian isotopy classes are canonically identified once we fix a diffeomorphism realizing the almost contact equivalence. Then performing a $(+1)$ -surgery along the loose Legendrian Λ_0 gives a contact manifold (Y, ξ') which is almost contact equivalent to (Y, ξ) . Theorem 1.1 implies that (Y, ξ') is overtwisted, therefore ξ is isotopic to ξ' .

Let M_b be the bottom half of the Weinstein cobordism M and M_t the Weinstein cobordism from (Y, ξ') to (S^{2n-1}, ξ_{st}) induced by the above $(+1)$ -surgery on Λ_0 . Then the glued cobordism $\tilde{M} := M_b \cup_Y M_t$ is a Weinstein cobordism from (S^{2n-1}, ξ_{ot}) to (S^{2n-1}, ξ_{st}) which is diffeomorphic to the smooth concordance M . \square

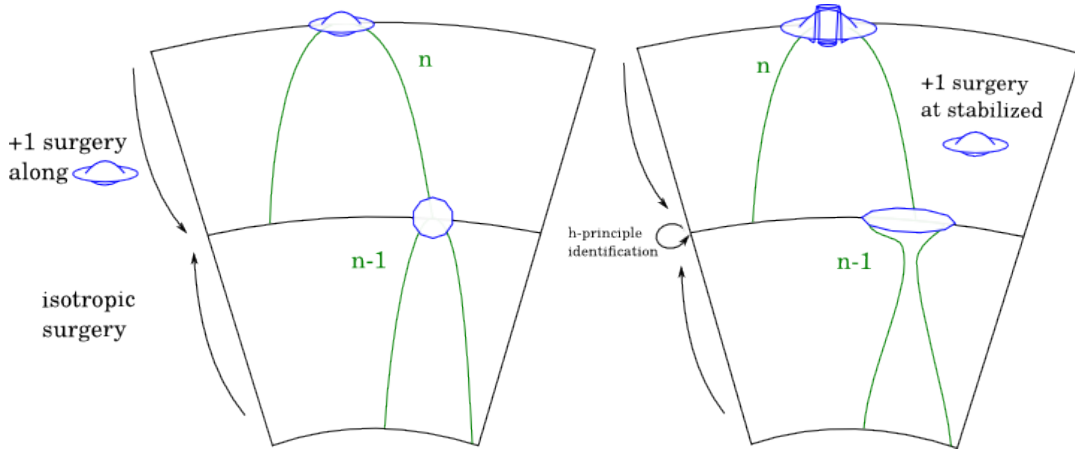


FIGURE 13. On the left, the symplectization of (S^{2n-1}, ξ_{ot}) with a cancelling pair of critical points. On the right, Weinstein concordance from (S^{2n-1}, ξ_{ot}) to (S^{2n-1}, ξ_{st}) obtained using $(+1)$ -surgery on a loose Legendrian.

Proposition 7.5 describes a strictly higher-dimensional phenomenon in contact topology. Indeed, it follows from the functoriality of Seiberg–Witten invariants that there exists no such Weinstein concordance in the case $n = 2$ [32], [27, Theorem 2.3], [29, Chapter 7]. The Weinstein concordance among spheres can now be glued to any Weinstein cobordism:

Proposition 7.5 implies the existence of all Weinstein cobordisms with an overtwisted concave boundary and arbitrary convex end which are not prohibited by topological restrictions:

Theorem 7.6. *Let (Y_-, ξ_{ot}) and (Y_+, ξ) be coorientable contact manifolds of dimension at least 5 and (Y_-, ξ_{ot}) overtwisted. Suppose there exists a smooth cobordism W from Y_- to Y_+ such that*

- a. *The relative homotopy type of W deformation retracts onto a half-dimensional CW complex.*

- b. W admits an almost complex structure J such that $J|_{Y_+}$ (resp. $J|_{Y_-}$) is homotopic through almost contact structures to ξ (resp. ξ_{ot}).

Then there exists a Weinstein cobordism (W, λ, φ) such that $\partial_-(W, \lambda) = (Y_-, \xi_{ot})$ and $\partial_+(W, \lambda) = (Y_+, \xi)$.

Proof. Let (Y^+, ξ_{ot}^+) be the overtwisted contact structure which is in the same almost contact homotopy class as ξ . The existence theorem for flexible Weinstein cobordisms [11] provides a flexible Weinstein structure (λ_f, φ_f) on W so that J is compatible with $d\lambda_f$ after homotopy, and $\partial_-(W, \lambda_f, \varphi_f) = (Y_-, \xi_{ot})$. Since (λ_f, φ_f) is flexible it follows that $\partial_+(W, \lambda_f)$ is overtwisted (Proposition 2.8). Also $\partial_+(W, \lambda_f, \varphi_f)$ is in the same almost contact homotopy class as ξ_{ot}^+ since both are homotopic to $J|_{Y_+}$, and therefore $\partial_+(W, \lambda_f, \varphi_f) \cong (Y_+, \xi_{ot}^+)$.

Now let $M = ([0, 1] \times S^{2n-1}, \lambda, f)$ be the Weinstein concordance defined in Proposition 7.5 and let $S(Y_+, \xi)$ be the symplectization of (Y_+, ξ) . Then $M \# S(Y_+, \xi)$ is a Weinstein cobordism which is diffeomorphic to $[0, 1] \times Y$, so that

$$\partial_- M \# S(Y_+, \xi) \cong (Y_+, \xi_{ot}^+) \text{ and } \partial_+ M \# S(Y_+, \xi) \cong (Y_+, \xi).$$

Gluing $(W, \lambda_f, \varphi_f)$ to $M \# S(Y_+, \xi)$ along $\partial_+(W, \lambda_f, \varphi_f) \cong (Y_+, \xi_{ot}^+) \cong \partial_- M \# S(Y_+, \xi)$ completes the construction. \square

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